Iterative Techniques and Application of Gauss-Seidel Method

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Abstract

This project work is based on `iterative techniques` by which the linear systems can be solved both analytically and numerically. In this project, we discussed Gauss-Seidel method in details. We have solved a linear problem numerically by Gauss-Seidel method. Finally we have discussed the error analysis of iterative method and also presented a conclusion on our calculations.

Key Words: Iterative techniques, Gauss-Seidel method, Linear system, Matrix.

1. Introduction

Iterative methods (or indirect methods) generate a sequence of approximations to the solution of a systems of algebraic equations. In contrast to direct methods, they do not produce the exact solution to the systems in a finite number of algebraic operations.

Many linear system are too large, to be solved by direct methods based on Gaussian elimination. For these systems, iteration methods are often the only possible method of solution, as well as being faster then elimination in many cases. The largest area for the application of iteration methods is the linear system arising in the numerical solution of partial differential equations. System of order 10^3 to 10^5 is not unusual, although almost all of the coefficients of the system will be zero. As an example of such problem is the numerical solution of Poisson's equation.

Besides being large, the linear system to be solved $A\underline{x}=\underline{b}$ often have several other important properties. They are sparse, which means that only a small percentage of the coefficient, are nonzero. The nonzero coefficient generally have a special pattern in the way they occur in A, and there is usually a simple formula that can be used to generate the coefficient a_{ij} as they are needed, rather then having to store them. As one consequence of these properties, the storage space for the vectors x and b may be a more important consideration then that of storage for A. The matrices A will often have special properties.

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We begin by defining and analyzing three classical iteration method; following that a general abstract frame work is presented for studying iteration method. The special properties of the linear system $A\underline{x}=\underline{b}$ are very important when setting up an iteration method for its solution.

Nomenclatures

a_{ij}	Matrix with a_{ij} as the entry in the *i*-th and *j*-th column	A	n x n Matrix
x	Column vector	**b**	Right hand side of matrix A
D	Diagonal matrix of A	U	Strictly upper triangular part of A
L	Strictly lower triangular part of A	$x^{(0)}$	Initial approximation
M_G	Gauss-Seidel matrix	$\|x\|$	Arbitrary norm of the vector **x**

2. General form of Iterative technique

An iterative technique to solve the *n* x *n* linear system $A\underline{x}=\underline{b}$ starts with an initial approximation $x^{(0)}$ to the solution *x*, and generates a the system sequence of vector $\{x^{(k)}\}_{k=0}^{\infty}$ that converges to *x*. Most of these iterative techniques involve a process that converts the system $A\underline{x}=\underline{b}$ into an equivalent system of the form $\underline{x}=B\underline{x}+C$ for some n x n matrix **B** and vector **C**.

Let the system be given by [Burden, R.L & Douglas Faires. J, 3rd Ed.]

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots\cdots+a_{2n}x_n=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3+\cdots\cdots+a_{3n}x_n=b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\quad+a_{nn}x_n=b_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots(1)$$

in which the diagonal elements a_{ii} do not vanish. If this is not the case, then the equations should be rearranged so that this condition is satisfied. Now, we rewrite the system (1) as

$$\left.\begin{array}{l} x_1=\frac{b_1}{a_{11}}-\frac{a_{12}}{a_{11}}x_2-\cdots-\frac{a_{1n}}{a_{11}}x_n \\ x_2=\frac{b_2}{a_{22}}-\frac{a_{21}}{a_{22}}x_1-\cdots-\frac{a_{2n}}{a_{22}}x_n \\ x_3=\frac{b_3}{a_{33}}-\frac{a_{31}}{a_{33}}x_1-\cdots-\frac{a_{3n}}{a_{33}}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n=\frac{b_n}{a_{nn}}-\frac{a_{n1}}{a_{nn}}x_1-\cdots-\frac{a_{n,n-1}}{a_{nn}}x_{n-1} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Suppose $x_1^{(1)}, x_2^{(1)}, \ldots\ldots x_n^{(1)}$, are any first approximations to the unknowns $x_1, x_2, \ldots\ldots x_n$ substituting in the right side of (2), we find a system of second approximations.

$$\left.\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_1^{(1)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
x_3^{(2)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \cdots - \frac{a_{3n}}{a_{33}} x_n^{(1)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly, if $x_1^{(n)}, x_2^{(n)}, \ldots\ldots x_n^{(n)}$ are a system of nth approximations, then the next approximation is given by the formula

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n)} - \cdots - \frac{a_{3n}}{a_{33}} x_n^{(n)} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n)}
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

If we write (2) in the matrix form

$$\underline{x} = \mathbf{B}\underline{x} + \mathbf{C} \ldots\ldots\ldots(5)$$

where **B** is the matrix of coefficients and **C** is the column vector of the right hand side.

Then the iteration formula (4) may be written as

$$\underline{x}^{(n+1)} = \mathbf{B}\underline{x} + \mathbf{C} \ldots\ldots\ldots(6)$$

3. Application of Iterative technique

i. Iterative techniques are seldom used for solving linear system of small dimension. Since the time required for sufficient accuracy exceeds that required for direct techniques such as the Gaussian elimination method.

ii. Iterative techniques are generally used for solving large system (spare matrix with maximum zero). In that case they are efficient in terms of computer storage and time requirement.

iii. These system arises mainly in the numerical solution of boundary value problems and partial differential equations.

iv. In computational applications we shall find it easier to use multiple indices to identify the unknowns; single indexing will be employed in discussions of the convergence of iterative methods.

4. Advantages

i. In general, one should prefer a direct method for the solution of a linear system but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods which preserve elements.

ii. Iterative methods generally require less computer storage and are easier to program than direct methods.

iii. For these systems, iteration methods are often the only possible method of solution, as well as being faster than elimination in many cases.

iv. Iterative methods are the methods of choice for solving certain types of linear partial differential equations using finite differences.

5. Classification

There are three methods for the iterative technique. They are:

i. Jacobi method (Method of simultaneous displacement)

ii. Gauss-Seidel method (Method of Successive displacement)

iii. SOR method (Successive over relaxation method)

Here only the Gauss-Seidel method is discussed in detail.

6. Gauss-Seidel method-derivation

It is a simple modification of the Jacobi method. This method uses an improved component as soon as it is available and it is called the method of successive displacements.

In this method, we use

$$x_i^{(k)} = (-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)} - \sum_{j=1+1}^{n} (a_{ij}\, x_j^{(k-1)}) + b_i)/\, a_{ii} \quad \text{for each } i = 1, 2, \ldots, n \qquad (7)$$

The system of equation $A\underline{x} = \underline{b}$ converges if A is strictly diagonally dominant.

6.1 Convergence of Gauss-Seidel method

$$D\underline{x}^{(k+1)} = L\underline{x}^{(k+1)} + U\underline{x}^{(k)} + \underline{b}$$

$$\Rightarrow (D-L)\underline{x}^{(k+1)} = U\underline{x}^{(k)} + \underline{b}$$

$$\therefore \underline{x}^{(k+1)} = (D-L)^{-1}\, U\underline{x}^{(k)} + (D-L)^{-1}\, \underline{b}$$

Here the iterative matrix is

$$M_G = (D-L)^{-1} U$$

M_G is called the Gauss-Seidel matrix.

Sufficient conditions for convergence are

i. Strictly diagonally dominant A.
ii. A is symmetric and positive definite.

6.2 Applications

i. The use of the most recently available information to update each component of x^k tends to increase the rate at which the sequence of estimates converges.

ii. Gauss- Seidiel methods are used with the linear system arising from solving some partial differential equations.

iii. It is an indirect method and it is used to solve a system of linear equations of the form, where $A\underline{x}=\underline{b}$ where A is n x n matrix.

6.3 Advantages

i. This is convenient for computer calculation, since the new value can be immediately stored in the location that held the old value, and this minimizes the number of necessary storage locations.

ii. Each new component $x_i^{(k+1)}$ is immediately used in the computation of the next component.

iii. The storage requirements for x with the Gauss-Seidel method is only half what it would be with the Gauss-Jacobi method.

iv. It is advantageous to use this method in the case of matrices having a large number of zero elements. It deserves to mention that the matrix should be diagonally dominant; that means the system of linear equations satisfy the condition $|a_{ij} / a_{ii}| < 1$, where a_{ij} are the elements of the corresponding matrix.

6.4 Disadvantage

i. This method converges about twice as fast as the Jacobi iterative method.
ii. Since it converges rapidly then its local error is no satisfied.

6.5 Solution of a problem by Gauss-Seidel Method

Example: Here we discussed the iterations of the Gauss-Seidel method for the following linear systems, using initial approximation $x^{(0)} = 0$.

$$
\begin{aligned}
4x_1 - x_2 \quad - x_4 \quad &= 0 \\
-x_1 + 4x_2 - x_3 - x_5 \quad &= 5 \\
-x_2 + 4x_3 \quad - x_6 &= 0 \\
-x_1 \quad + 4x_4 - x_5 &= 6 \\
-x_2 \quad - x_4 + 4x_5 - x_6 &= -2 \\
-x_3 \quad - x_5 + 4x_6 &= 6
\end{aligned}
$$

Solution

The linear system $A\underline{x} = \underline{b}$ given by

$$
\begin{aligned}
E_1 &: \; 4x_1 - x_2 \quad - x_4 = 0 \\
E_2 &: \; -x_1 + 4x_2 - x_3 - x_5 = 5 \\
E_3 &: \; -x_2 + 4x_3 \quad - x_6 = 0 \\
E_4 &: \; -x_1 \quad + 4x_4 - x_5 = 6 \\
E_5 &: \; -x_2 \quad - x_4 + 4x_5 - x_6 = -2 \\
E_6 &: \; -x_3 \quad - x_5 + 4x_6 = 6
\end{aligned}
$$

To solve the above system by Gauss-Seidel method.

To convert $A\underline{x} = \underline{b}$ to the form $\underline{x} = T\,\underline{x} + C$, solve equation E_i for x_i, for each i = 1,2,3,4,5,6 to obtain

$$x_1 = \frac{1}{4}x_2 + \frac{1}{4}x_4$$
$$x_2 = \frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{1}{4}x_5 + \frac{5}{4}$$
$$x_3 = \frac{1}{4}x_2 + \frac{1}{4}x_6$$
$$x_4 = \frac{1}{4}x_1 + \frac{1}{4}x_5 + \frac{6}{4}$$
$$x_5 = \frac{1}{4}x_2 + \frac{1}{4}x_4 + \frac{1}{4}x_6 - \frac{2}{4}$$
$$x_6 = \frac{1}{4}x_3 + \frac{1}{4}x_5 + \frac{6}{4}$$

where

$$T = \begin{bmatrix} 0 & 1/4 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 1/4 & 0 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 0/4 \\ 5/4 \\ 0/4 \\ 6/4 \\ -2/4 \\ 6/4 \end{bmatrix}$$

For an initial approximation let $x^{(0)} = (0,0,0,0,0,0)^t$ and generate $x^{(k)}$ by

$$x_1^{(k)} = 0.25\,x_2^{(k-1)} + 0.25\,x_4^{(k-1)}$$
$$x_2^{(k)} = 0.25\,x_1^{(k-1)} + 0.25\,x_3^{(k-1)} + 0.25\,x_5^{(k-1)} + 1.25$$
$$x_3^{(k)} = 0.25\,x_2^{(k-1)} + 0.25\,x_6^{(k-1)}$$
$$x_4^{(k)} = 0.25\,x_1^{(k-1)} + 0.25\,x_5^{(k-1)} + 1.5$$
$$x_5^{(k)} = 0.25\,x_2^{(k-1)} + 0.25\,x_4^{(k-1)} + 0.25\,x_6^{(k-1)} - 0.5$$
$$x_6^{(k)} = 0.25\,x_3^{(k-1)} + 0.25\,x_5^{(k-1)} + 1.5$$

generate $x^{(1)}$ by

$$x_1^{(1)} = 0.25x_2^{(0)} + 0.25x_4^{(0)} = 0$$
$$x_2^{(1)} = 0.25x_1^{(0)} + 0.25x_3^{(0)} + 0.25x_5^{(0)} + 1.25 = 1.25$$
$$x_3^{(1)} = 0.25x_2^{(0)} + 0.25x_6^{(0)} = 0.3125$$
$$x_4^{(1)} = 0.25x_1^{(0)} + 0.25x_5^{(0)} + 1.5 = 1.5$$
$$x_5^{(1)} = 0.25x_2^{(0)} + 0.25x_4^{(0)} + 0.25x_6^{(0)} - 0.5 = 0.1875$$
$$x_6^{(1)} = 0.25x_3^{(0)} + 0.25x_5^{(0)} + 1.5 = 1.625$$

Additional iterates, $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t$, are generated in a similar manner and are presented in Table 1

Table 1- Values of x's by Gauss-Seidel method

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$	$x_6^{(k)}$
1	0	1.25	0.3125	1.5	0.1875	1.625
2	0.6875	1.546875	0.79296875	1.71875	0.7226525	1.87890625
3	0.81640625	1.833007813	0.927978515	1.884765625	0.89916992	1.95678711
4	0.9294433	1.93917949	0.973983764	1.95715332	0.96327209	1.98431397
5	0.97407532	1.977832794	0.990536689	1.984336853	0.98662090	1.99428940
6	0.99054241	1.991925001	0.996553599	1.994290828	0.99512631	1.99791998
7	0.99655396	1.997058466	0.99874461	1.997920066	0.99822463	1.99924231
8	0.99874463	1.998928467	0.999542694	1.999242315	0.99935327	1.99972399
9	0.99954270	1.999609665	0.999833414	1.999723992	0.99976441	1.99989946
10	0.99983341	1.99985781	0.999939316	1.9999899457	0.99991418	1.99996337
11	0.99993932	1.999948203	0.999977894	1.999963374	0.99996874	1.99998666
12	0.99997789	1.999981131	0.999921947	1.999986658	0.99998861	1.99999514

Calculation

The decision to stop after twelve iteration is based on the following calculations

$$\frac{\left\| x^{(10)} - x^{(9)} \right\|}{\left\| x^{(10)} \right\|} = \frac{\|1.99996337 - 1.99989946\|}{\|1.99989946\|} = 3.1957\times 10^{-5} < 10^{-3}$$

and
$$\frac{\left\| x^{(12)} - x^{(11)} \right\|}{\left\| x^{(12)} \right\|} = \frac{\|1.99999514 \quad 1.99998666\|}{\|1.99999514\|} = 4.2400\times 10^{-6} < 10^{-3}$$

These calculation shows that $x^{(12)}$ is accepted as a reasonable approximation to the solution.

Error calculation

According to our problem we only got approximate values. For exact values we can use Mathmetica [a computer package]. In Mathematica we use the following commands for exact value:

Solve [{E_1, E_2, E_3, E_4, E_5, E_6},{x_1, x_2, x_3, x_4, x_5, x_6}]

After execution it gives the exact values as

$$x_1 = 1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 2,\ x_5 = 1,\ x_6 = 2$$

Now we calculate the error and the percentage of error for every x^i as

$$\text{Error} = |Exact\ value - Approximate\ valu|$$

$$= \left| x_1 - x_1^{(12)} \right| = |1 - 0.99997789| = 0.00002211$$

$$\text{Percentage of error} = \frac{\left(|Exact - Approximate| \times 100\right)}{Exact}$$

$$= \frac{0.00002211\times 100}{1} = 0.002211\ \%$$

In a similar way, the percentage of error for other values of x_i are calculated and presented in Table 2.

Table2- Calculational errors in Gauss-Seidel method

x_i	Exact value	Calculated value	Error %
x_1	1	0.99997789	0.002211
x_2	2	1.999981131	0.00094345
x_3	1	0.999921947	0.0078053
x_4	2	1.999986658	0.0006671
x_5	1	0.99998861	0.001139
x_6	2	1.99999514	0.000243

7. Conclusion

In this project work we have studied the Gauss-Seidel methods. Then we solved a problem numerically by the Gauss-Seidel method. This method is convenient for computer calculation, since the new value can be immediately stored in the location that held the old value, and this minimizes the number of necessary storage. Each new components is immediately used in the computation of the next component. In this method we got approximate values accurate to 10^{-4}. We have calculated the absolute and percentage of errors which are presented in the tabular form. We have also calculated the exact values by using Mathematica. From the table we thus observe that the percentage of errors are of the order 10^{-4} . This indicates that the numerical method which we have adopted gives a very good approximation of the exact solutions. We can therefore conclude that this method is suitable for finding solutions of linear systems numerically having large number of unknowns.

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