Fractals

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Abstract

A fractal is an object or quantity that displays self-similarity, in a somewhat technical sense, on all scales. The object need not exhibit exactly the same structure on all scales, but the same "type" of structures must appear on all scales. A plot of the quantity on a log-log graph versus scale then gives a straight line, whose slope is said to be the fractal dimension. The prototypical example for a fractal is the length of a coastline measured with different length rulers. The shorter the ruler, the longer the length measured, a paradox known as the coastline paradox. The Mandelbrot set is a famous example of a fractal. A fractal is generally "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole,"a property called self-similarity. Roots of mathematical interest on fractals can be traced back to the late 19th Century; however, the term "fractal" was coined by Benoît Mandelbrot in 1975 and was derived from the Latin fractus meaning "broken" or "fractured."

Key words: Fractal, souse, mandelbrot, geometric shape, broken.

Introduction

A mathematical fractal is based on an equation that undergoes iteration, a form of feedback based on recursion. A fractal often has the following features:

- It has a fine structure at arbitrarily small scales.
- It is too irregular to be easily described in traditional Euclidean geometric language.
- It is self-similar (at least approximately or stochastically).
- It has a Hausdorff dimension which is greater than its topological dimension (although this requirement is not met by space-filling curves such as the Hilbert curve).
- It has a simple and recursive definition.

Because they appear similar at all levels of magnification, fractals are often considered to be infinitely complex (in informal terms). Natural objects that approximate fractals to a degree include clouds, mountain ranges, lightning bolts, coastlines, snow flakes, various vegetables (cauliflower and broccoli), and animal coloration patterns. However,

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not all self-similar objects are fractals—for example, the real line (a straight Euclidean line) is formally self-similar but fails to have other fractal characteristics; for instance, it is regular enough to be described in Euclidean terms. Images of fractals can be created using fractal-generating software. Images produced by such software are normally referred to as being fractals even if they do not have the above characteristics, as it is possible to zoom into a region of the image that does not exhibit any fractal properties.

Bounded and Unbounded set

A set in R^2 is called bounded if it can be enclosed by a suitably large circle and otherwise it is called unbounded (Figure 1).

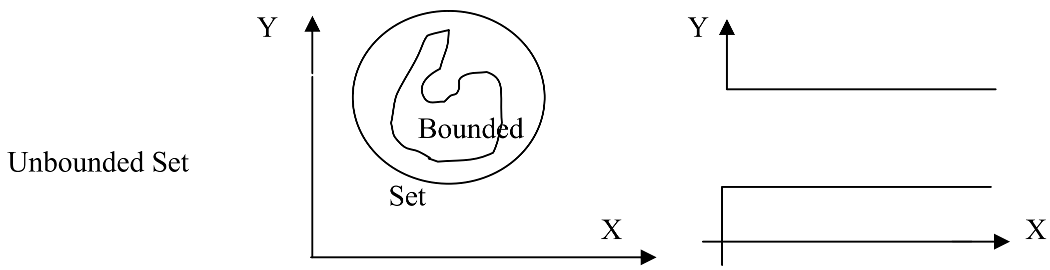


Figure: 1

Closed set

A set in R^2 is called closed if it contains all of its boundary points (Figure 2).

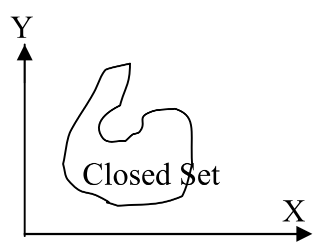


Figure: 2

The boundary points lie in the set.

Congruent Set

Two sets in R^2 are called congruent if they can be made to coincide exactly by translating and mutating them appropriately within R2 (Figure 3).

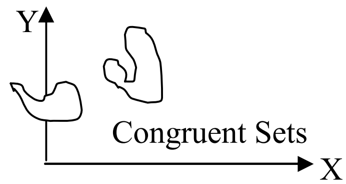


Figure: 3

Dilation of Set

If T: R^2 -----> R^2 is the linear operator that scales by a factor of S and if Q is a set in R^2, then the set T(Q) (the set of images of points in a under T) is called a dilation of the set Q if S>I and a contraction of a if O<S<I (Figure 4). In either case we say that T(Q) is the set Q scaled by the factor S.

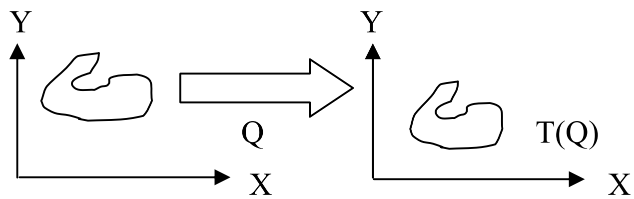


Figure: 4

Self- similar set

A closed and bounded subset of the Euclidean plane R^2 is said to be self -similar if it can be expressed in the form

$$S = S_1 \cup S_2 \cup S_3 \cup \ldots\ldots S_k$$

Fractal dimension

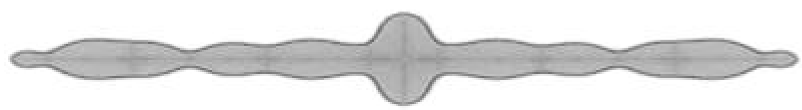
The dimension of a fractal curve is a number that characterizes the way in which the measured length between given points increases as scale decreases. Whilst the topological dimension of a line is always 1 and that of a surface always 2, the fractal dimension may be any real number between 1 and 2. The **fractal dimension** *D* is defined by

$$D = \frac{\log(L2 / L1)}{\log(S1 / S2)} \qquad \ldots \ (1)$$

where L_1, L_2 are the measured lengths of the curves (in units), and S_1, S_2 are the sizes of the units (*i.e.* the scales) used in the measurements.

Base-Motif Fractals

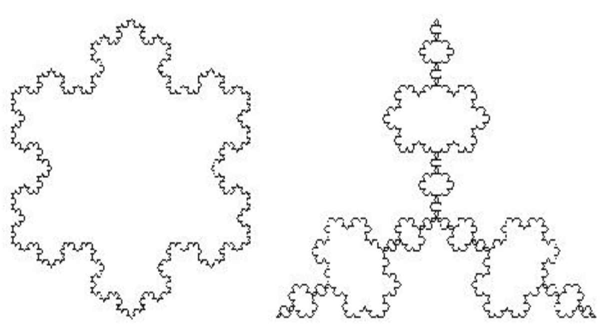
Take a shape — any shape that is composed of line segments. Call this shape a *base*. Now take another shape and call it the *motif*. Substitute every line segment in the base with the motif. Do the same with the resulting figure. Now continue substituting an infinite amount of time. The process you are doing is called **generator iteration**. What you get at the end is a *base-motif fractal*.



Three things effect a base-motif fractal:

i) Shape of the base — the most common bases are a line segment, a square, and an equilateral triangle.
ii) Shape of the motif — this obviously has the greatest effect on the outcome. There is an infinitely Great Variety of motifs.
iii) Positioning of the motif — if the base is a square or a triangle, you can place the motif facing either inside or outside, which will result in different fractals.

For example, by using the following base and motif you can get two different fractals. In the **Koch Snowflake**, the motif is facing outside, while in the **Koch Antisnowflake**, it is facing inside:

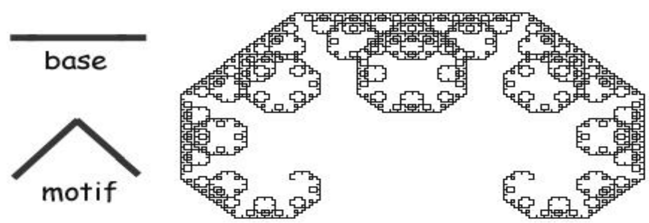


If you position the motif the same way all the time during **iteration**, you will get a regular base-motif fractal. If you position it differently in some places, you will get a special type of a base-motif fractal called a **sweep**. Base-motif fractals are also categorized depending on their **fractal dimension**. When the dimension is between 1 and 2, we call it a *fractal curve*. When it is less than 1 it is called a **dust** and when it is equal to 2 it is called a **Peano Curve**.

Examples

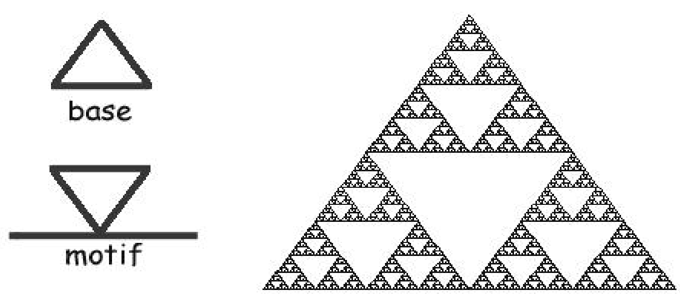
Below are three examples of base-motif fractals that use different bases, motifs, and positioning:

Levy Curve



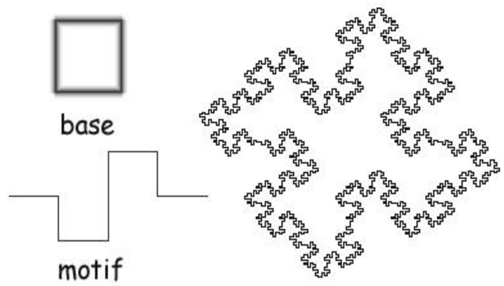
Positioning: there is only one way to position a motif if the base is a line segment.

Sierpinski Triangle



Positioning: inside

Koch Island



Positioning: outside

Uses

Because base-motif fractals are so perfectly shaped, they cannot be very useful in modeling real-life things. Yet, they can create simple models of **coastlines and borderlines**. Recently, they were also used to create relatively good models of **economy**.

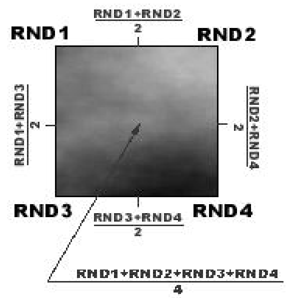
Fractal geometry that deals with objects in non-integermetry is a **Plasma Fractal**.

Plasma fractals are perhaps the most useful fractals there are. Unlike most other fractals, they have a random element in them, which gives them **Brownian self-similarity**. To create a plasma fractal on a rectangular piece of plane, you do the following:

i) Randomly pick values for the corners of the rectangle

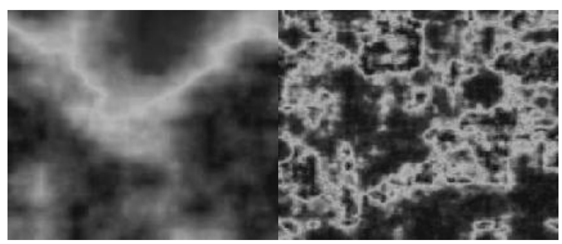
ii) Calculate the value for the center of the rectangle by taking the average of the corners and adding a random number multiplied by a preset roughness parameter.

iii) Calculate the midpoints of the rectangles' sides by taking averages of the two nearest corners and adding a random number multiplied by the roughness parameter.



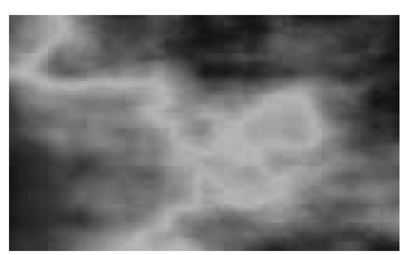
iii) You now have four smaller rectangles. Do steps ii-iv for each of them.

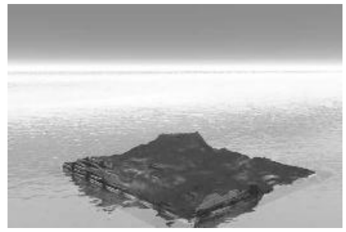
After calculating the values, you can draw the picture by coloring the points depending on their value. Below are two plasma fractals with a small and a large roughness value:



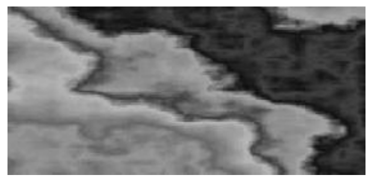
Uses

Due to their randomness, plasma fractals closely resemble nature. Because of this, you can use plasma fractals in many different applications. For example, by using standard atlas colors and determining the heights by the values of the points, we can get very **realistic landscapes**:



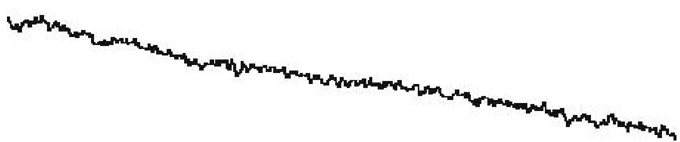


These kinds of fractal landscapes were used for **special effects** in many movies, including Star Trek. By using other colors, we can get realistic pictures of **clouds**:



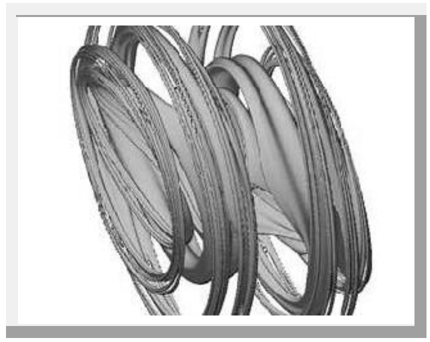
Plasma Fractals in One Dimension

We can create equivalents of plasma fractals in 1 dimension by randomly choosing the endpoints and then calculating the midpoint using roughness. This allows us to create pictures like the one below:



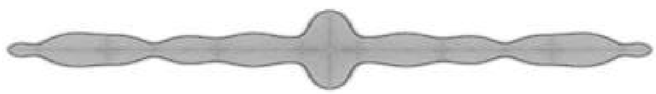
Quaternions

Complex numbers, in the form a + bi are essential in creating fractals like **Julia Sets**, which use the complex number formula $z = z^2 + c$. There are, however, equivalents of complex numbers which contain 4 instead of 2 terms and can be expressed as a + bi + cj + dk. They are called *quaternions* and were introduced by Hamilton in 1847. There is a completely separate algebra of quaternions, which is very important in math and physics. Quaternions can also be used to create fractals equivalent to Julia Sets by being used in the formula $z = z^2 + c$ instead of complex numbers. Since quaternions have 4 terms, the fractal is 4-dimensional. We manage to display them by taking 3D "slices" of the fractals at certain places. The following picture is an example of quaternion:



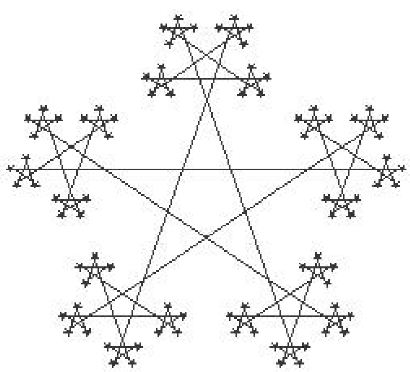
ellipses, circles etc. Fractal geometry, however, is described in ala set **Star Fractals.**

Star fractals is another easy-to-make type of fractals. To make a star fractal, you start with some geometric figure (usually a polygon). You then attach smaller versions of that figure to every corner. Then even smaller versions of the figure are attached to the corners and this type of **iteration** is continued.

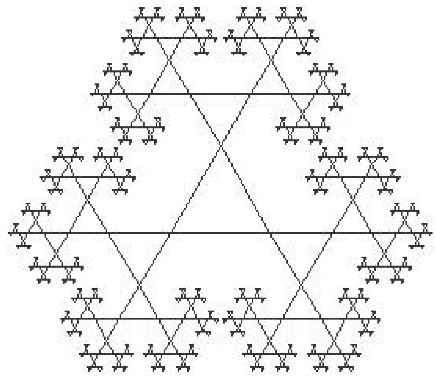


Different kinds of stars

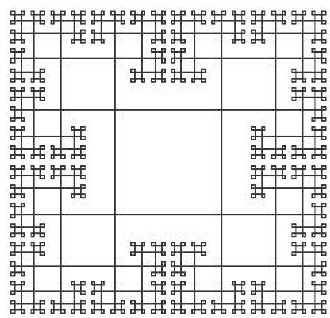
The most famous representative of this type of fractals is the **Star Fractal**, which uses a five-corner star for the original figure:



Other star fractals include the one that uses a triangle:



And the square star which uses a square:



Hausdorff Dimension

Informally, self-similar objects with parameters *N* and *s* are described by a power law such as

$$N = s^d.$$

where

$$d = \frac{\ln N}{\ln s}$$

is the "dimension" of the scaling law, known as the Hausdorff dimension.

In 1919 the German mathematician Felix Hausdorff (1868-1942) gave an alternative definition for the dimension of an arbitrary set in R_n.

The Hausdorff dimension of a self-similar set s of the form

$S = S_1 \cup S_2 \cup S_3 \cup ....S_k$; where $S_1, S_2, ...S_k$ are non over lapping sets.

It is denoted by $d_H(S)$ and is defined by $d_H(S) = \dfrac{\ln(k)}{\ln(1/s)}$. where " ln " denotes the natural logarithm function.

Formally, let A be a subset of a metric space X. Then the Hausdorff dimension D(A) of A is the infimum of $d \geq 0$ such that the d-dimensional Hausdorff measure of A is 0 (which need not be an integer).

In many cases, the Hausdorff dimension correctly describes the correction term for a resonator with fractal perimeter in Lorentz's conjecture. However, in general, the proper dimension to use turns out to be the Minkowski-Bouligand dimension (Schroeder 1991).

Julia Sets

The world as we know it is made up of objects which exist in integer dimensions, single dimensional points, one dimensional lines and curves, two dimension plane figures like circles and squares, and three dimensional solid objects such as spheres and cubes.

Gaston Julia studied rational polynomial expressions of various degrees (e.g., $z^4 + z^3/(z + 1) + z^2/(z^3 + 4z^{2+}5) + c$), but in this presentation I will limit the discussion mostly to the family of sets generated by the special quadratic case form $f(z) = z^2 + c$. Here z represents a variable of the form a+ib (a and b real numbers) which can take on all values in the complex plane. The quantity c also is defined as a complex number, but for any given Julia set, it is held constant (thus it is termed a parameter). In other words, there are an infinite number of Julia sets, each defined for a given value of c, though the ones with smaller values of c (i.e., $|c| < \sim 2$) are particularly interesting graphically.

Used once, the simple expression $f(z) = z^2 + c$ has little potential to create anything interesting--it is only by repeatedly iterating it that the Julia set can be defined. When the output of the expression f(z) is fed back into the expression as a new value of z, this is called iteration, a type of feedback process. Thus, for any n:

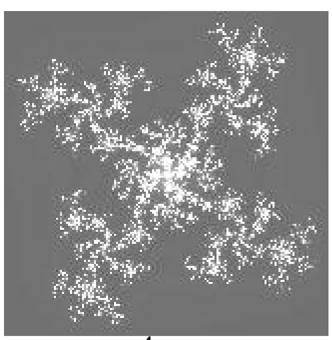
$$z_{n+1} = f(z) = z_n^2 + c$$

and each new computed value of f(z) becomes the subsequent input value of z via the feedback loop.

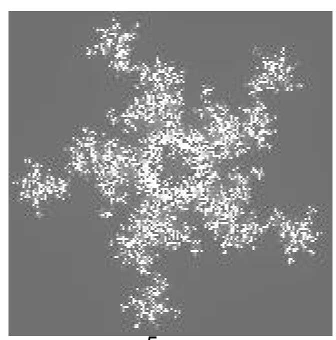
Note that authors typically express purely real values of c such as -1+0i as "-1", and this can confuse the uninitiated who may not realize that c is still defined in the complex plane. For the special case where c=0+0i, the Julia set is simply a (nonfractal) circle with radius 1. A nice geometric explanation of the result of squaring a complex number and adding a complex constant to it is given at the Chaos Hypertextbook website (Elert 22.shtml). For any z, this transformation consists of a contraction (for $|z|<1$) or dilation (for |z|>1) resulting from multiplying by |z|, as well as a doubling of the polar angle (i.e., the argument) of z, and then a translation by c.

For any given starting value of z, say z_0, there are two possibilities for what will happen to the iterated values of f(z) as n increases toward infinity: either f(z) can continue to grow without bounds or it will stay bounded. Points z_0 in the complex plane that do not stay bounded with successive iterations of f(z) are said to be in the escape set E_c. All other points in the complex plane stay bounded as n is taken to infinity--they are termed prisoners and are said to be in the prisoner set P_c defined for a given c. All points must either be in one or the other set. The common boundary between the escape set and the prisoner set is called the Julia set J_c, defined for a particular value of c. The threshold radius r(c) = max(|c|, 2) provides a useful test criterion for computer implementation. If an orbit z_k ever exceeds the threshold radius r(c), it is certain that the orbit will escape toward infinity and therefore the starting point is in the escape set (Peitgen et al. 794).There seems to be a little definitional confusion in the literature as to whether the boundary points (i.e., the Julia set) are themselves part of the prisoner set, but this must be the case. The complex plane is divided solely into prisoner and escaping points, so the boundary points must belong to P_c since they cannot belong to E_c (otherwise they would escape under repetitive iterations, which is not the case). There are Julia sets (i.e., of boundary points) which do not enclose any interior prisoner points. Since P_c is by definition what remains of the complex plane after removing E_c, the boundary and prisoner points must coincide. An example of such a set is J_c for c=0+i (Peitgen et al. 798). I have not seen this explicitly stated but suspect that there are no Julia sets having any escaping points which are completely surrounded by prisoner points.

As mentioned above, Julia sets can also be formed from higher degree and more complex expressions. The following are Julia sets for the iterated functions f (z) = z^4 + c and f (z) = z^5 + c, respectively:



$z^4 + c$
c = 0.6+0.55i

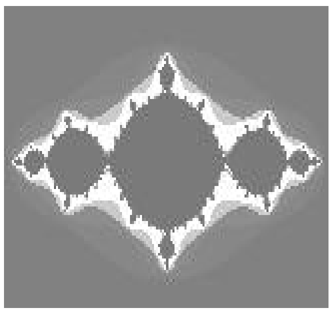


$z^5 + c$
c = 0.8+0.6i

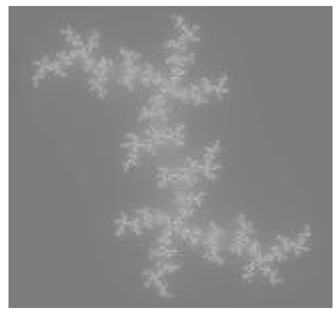
Invariance and Symmetry of the Julia Sets

The points of a Julia set (i.e., the boundary points) are "invariant" with respect to further iterations of f(z). Here, invariance does not imply that $f(z_j) = z_j$, but that, for any point z_j belonging to the set J_c, $f(z_j)$ is also a member of J_c (Peitgen et al. 822).

Julia sets with purely real c (i.e., of the form c = a+0i) are reflection symmetric (that is, the part of the set below the real axis may be derived by reflecting the part above the real axis). Sets with complex c exhibit rotational symmetry--i.e., there is an axis passing through the origin which divides the set into parts which, when rotated 180 degrees, will coincide (Elert 22.shtml).



A reflectionally symmetrical Julia set
c = -1+0i



A rotationally symmetrical Julia set
c = 0.3 + 0.6i

Connected versus Disconnected Julia Sets

Depending on the value of c selected, the resultant Julia set may be connected or disconnected--in fact, either totally connected or totally disconnected. There are several types of mathematical connectedness. In the case of Julia sets, the type of connectedness referred to is "pathwise" connectedness, meaning that one can trace a path from a point in the set to other points in the set without leaving the set (Gagliardo). However, the type of connectedness referred to is less clear in Peitgen et al (803) and may not be pathwise. Connected Julia sets are "completely connected" as opposed to being merely "locally connected", a result shown independently by Julia and by Fatou (Gagliardo). Topologically, connected Julia sets are either equivalent to a severely deformed circle or to a curve with an infinite series of branches and sub-branches called a dendrite (e.g., the Julia set for c=0+i)" (Elert 22.shtml). Note that graphical displays of connected Julia sets often appear to demonstrate separate subsets even though they are in fact connected. When c is on the real axis (c = a+0i), the Julia sets are connected only for the x-interval [-2, 1/4] (Peitgen et al. 832).

Mandelbrot called disconnected Julia sets a "dust" of points (Mandelbrot 79), or "Fatou dust" (after Pierre Fatou 1878-1929) (Mandelbrot 182). This is a logical term, since a disconnected Julia set consists of individual points in the complex plane which, like sparsely sprinkled dust on a sheet, are not connected to any others. Another term used to describe a disconnected Julia set is *Cantor dust* (Peitgen et al. 798). The Cantor set of points is a totally disconnected set produced by successively dividing the line segment [0,1] in thirds and discarding the center segment yielding [0,1/3] and [2/3, 1], then repeating for each remaining line segment ad infinitum (Peitgen el al 68). The distribution of points in a disconnected Julia set qualitatively resembles the appearance of the more easily envisioned Cantor dust in that they are totally disconnected. Elert states "Disconnected sets are completely disconnected into a countably infinite assembly of isolated points. In addition, these points are arranged in dense groups such that any finite disk surrounding a point contains at least one other point in the set." (Elert 22.shtml).

The criterion which determines whether a Julia set is connected or disconnected will be discussed below with the Mandelbrot set.

Are Julia Sets Fractals? Dimensionality

The definition of fractal is inextricably connected to the concept of fractal dimension. We are all familiar with the topological dimension in describing the dimensionality of an object. A point has a topological dimension of 0, a curve or straight line a topological dimension of 1, a smooth surface a dimension of 2, and a smoothly demarcated solid object a dimension of 3. Even if curves, surfaces, or solids have rough or irregular edges, they may be deformed topologically to yield smooth objects with the stated topological dimensions. However, the fractal (non-topological) dimension of fractals (such as the Cantor dust of points or the Sierpinski gasket) incorporates the concept that their infinite ramifications in effect cover more of their Euclidean space than their topological dimension would suggest. For example, the well known Koch curve is a space-filling curve, a single line of infinite length and infinite angularity, which has topological dimension of 1 but a fractal dimension of log 4 / log 3 = c. 1.26. Speaking intuitively, it is able to cover more than the infinitesimal amount of the plane which a traditional one-dimensional curve can cover.

The exact definition of what constitutes a fractal seems to be undecided. Mandelbrot defined a fractal in 1977 as "a set in metric space for which the Hausdorff. Besicovitch dimension D exceeds the topological dimension D_T", but also stated that this rigorous definition should be considered tentative. He also offered a revised definition: "a set for which Frostman capacitary dimension > topological dimension" (Mandelbrot 361-2). In fact he believes that "one would do better without a definition." The topological dimension for the complex plane is simply 2. However, the choice of which of many fractal dimensional metrics is to be compared with the topological dimension is a controversial and complex subject.

Incidentally, it is often stated that a fractal has a fractional (nonintegral) dimension (the name *fractal* might seem to imply this), and this is usually the case, particularly regarding natural nonmathematically generated fractal objects. But some mathematical fractals have integer dimensions that exceed their topological dimensions. For example, Mandelbrot lists the path of Brownian motion ($D_T = 1$, $D = 2$) and the Lebesque-Osgood monster surface ($D_T = 2$, $D = 3$) (Mandelbrot 446-7).

Computer simulations verify the boundary of many of the Julia sets to be infinitely complex and never smooth regardless of the magnification, and thus such sets satisfy Mandelbrot's intended etymological root for *fractal*, i.e., having a fractured or broken contour-- the Latin adjective *fractus* derives from the verb *frangere* meaning "to break" or create irregular fragments (Mandelbrot 4). The theoretical computation of the fractal dimension of a Julia set is apparently not a straightforward calculation and of course dependent on the dimension metric utilized and the parameter c. For instance, the 1992 textbook by Peitgen et al. does not include an estimate of the fractal dimension for Julia sets, though it does for the other strange (fractal) attractors described such as the Hénon (D=1.28), Lorenz (D=2.073), and Rössler (c. 2.01-2.02) attractors. Elert used the Macintosh program Fractal Dimension Calculator (Bourke), which I have not been able to test. He calculated the empiric fractal dimension of a specific Julia set, that for c = -1+0i, and estimates the value at 1.16 (Elert 33.shtml). Obviously this result is at great variance with respect to Mitsuhiro's.

Self-similarity

Fractals often exhibit self-similarity. Strict similarity is defined mathematically by Mandelbrot (Mandelbrot 349), paraphrased as follows: Given a real ratio parameter r, a positive integer N, and a bounded set S of points $x = (x_1, x_2, x_3, ..., x_E)$ defined in a Euclidean space of dimension E, then S is self-similar if it is the union of N nonoverlapping subsets each of which is congruent to rS. Congruent means that they coincide, if necessary after a suitable rotation and displacement. This is readily observed in such regularly constructed and strictly self-similar objects as the Cantor set and the Koch curve (Peitgen et al. 76, 204).

"[Suitably] selected fragments of a Julia set are strictly [self]-similar to the set as a whole." (Elert 23.shtml). In contrast, "fragments of the Mandelbrot [set] are only quasi-similar to the set as a whole. Furthermore, the motif of this quasi-self-similarity varies from one region to another and from one level of magnification to another." (Elert 23.shtml). Mandelbrot wished his own set, i.e., the "Mandelbrot Set", to be included as a fractal, so he introduced the concept of statistical self-similarity and self-affinity and other definitional extensions to include a broader range of objects that appear similar to the eye (Mandelbrot 350).

Other expected fractal properties, such as sensitivity, mixing, and periodicity are discussed below for Julia sets.

Fixed Points and Strange Attractors

Any value of c is associated with fixed points for z, namely the two (noniterative) solutions of the equation $z^2 - z + c = 0$. (This discussion of fixed points does not apply to the disconnected sets, about which I have found no information.) Taking as an example the connected set for c=-0.5+0.5i, the solutions to this quadratic equation are given by

$z = 1/2 \pm (\mathrm{sqrt}(1 - 4c))/2$, or

$z = 1/2 \pm 1/2\mathrm{sqrt}((3 - 2i))$, or

$z_1 = 1.4087-0.2751i$

$z_2 = -0.4087+0.2751i$.

For many values of c producing connected Julia sets (i.e., within the cardioid, see below), successive iterations of f(z) are found to converge to a single limiting z value = z_2. Moreover, the orbits of other starting points in the interior prisoner set (i.e., not in the Julia set) when iterated converge to this fixed point. Thus z_2 is called an attracting fixed point and the interior prisoner set lies in the basin of attraction for this point. (Incidentally, the escape set of points for a Julia set may be said to lie in the basin of attraction of the point at infinity.)

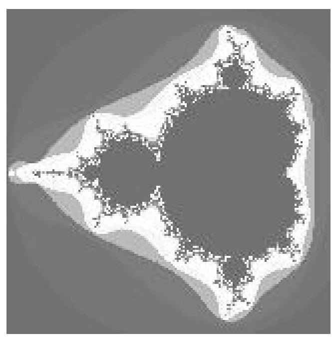
The other fixed point z_1 behaves differently than z_2. It is said to be a point on the Julia set and, provided sufficient calculation accuracy were available, iterations of this value are said to yield other points on the Julia set. Thus this fixed point is said to be a repelling fixed point. Peitgen elaborate that fixed points are repelling when the complex derivative at that point has absolute value >1, attracting when <1, and "indifferent" when =1 (Peitgen 822-823).

Objects like the Julia sets act as strange attractors, a term coined by Ruelle and Takens and currently lacking a precise definition (Peitgen 657, 671). The word "strange" connotes the infinitely complex and self-similar fractal nature of the attractor. For example, the Hénon attractor is comprised of an infinite number of nearby but nonoverlapping curves, as are the Rössler and Lorenz attractors. A Julia set is an attractor in the sense that values of z belonging to J_c when further iterated continue to produce other values lying in J_c. That is, the set seems to attract orbits beginning in the set. Julia sets also act as attractors for the prisoner and escape sets under the inverse transformation, see below. Like other chaotic systems (Peitgen 536), Julia sets exhibit (i) strong sensitivity to initial conditions (i.e., the orbits of nearby points in J_c rapidly spread over the set with successive iterations) and (ii) strong mixing inasmuch as the orbits of even a short interval of adjacent points on the Julia Set will, with successive iterations, spread out to quickly cover the entire Julia set (Peitgen et al. 824). The third expected criterion which defines chaotic behavior is periodicity, and this phenomenon is further described below.

The concept that a Julia acts as a strange attractor may be further illustrated as follows. Peitgen et al. (821, 229-296) introduce the useful concept of what they call the "Multiple

Copy Reduction Machine" (their name for an Iterated Function System). This is a sort of fancy and purely virtual photocopying machine which is capable of applying repeated nonlinear transformations to starting images according to a particular transformation formula. For Julia sets, they utilize transformation functions derived from the equation $w = z^2 + c$ to compute the "preimage" for any given image, namely w --> ±sqrt(w - c). A preimage P of an image I is the set of points which yields I when a transformation such as $z^2 + c$ is applied to P. It is therefore obtained by the inverse transformation of $z^2 + c$, the transformation which is usually said to define Julia sets. When we iterate $z^2 + c$ for most connected Julia sets, the orbits of most starting points z_0 in the z plane either tend toward the "fixed point" at infinity or toward the interior fixed point (exceptions include the Julia set points J_c, which are invariant). When the inverse expression for the transformation is iterated, the interior points of the prisoner set orbit outward and converge on the Julia set boundary, and similarly the exterior points in the escape set orbit inward and converge on the Julia set boundary. Therefore, all starting points of the plane are either already on or converge toward the Julia set in the limit under the inverse transformation. The Julia set seems in this sense to attract the orbits of all points of the z plane and thus may be termed an attractor. The iteration of the inverse transformation provides an alternative algorithm for computer generation of images of Julia sets.

The Relationship Between Julia Sets J_c and the Mandelbrot Set M



The Mandelbrot Set M

Please note that it is my intention to focus on Julia sets, so that I will present only limited comments on the properties of the Mandelbrot set, primarily as it relates to Julia sets.

Mandelbrot Set M

The Mandelbrot set M, discovered by Benoit B. Mandelbrot c. 1979, can be defined as the set of all values of the parameter c (Mandelbrot uses the character μ) for which the corresponding Julia sets are each connected, in fact totally connected. Alternatively, it can be defined as the set of values of c for which the orbits (successive iterations) of z_0 = 0+0i remain bounded (Mandelbrot 183, Peitgen et al. 843). The point 0+0i is termed the critical point for Julia sets. This simple test, i.e., the boundedness of iterations of 0, thus determines whether a Julia set is connected, a result discovered independently by Julia and by Fatou (Gagliardo). The definition relating to iterations of 0 was the one Mandelbrot actually used in his initial explorations of M (Mandelbrot 183). M is a connected set, as shown by Douady and Hubbard (Peitgen et al. 849). However, this statement must apparently be qualified to "at least locally connected", since there are said to be an infinite number of points in M that are not currently known to be connected (Brin mandel_why.html).

Planes in Which Julia and Mandelbrot Sets are Defined

The definition of the prisoner set of a Julia set, defined as the P_c such that iterations of z --> z^2 + c remains bounded, is similar to the second definition of the Mandelbrot set, where iterations of c --> c^2 + c --> $(c^2 + c)^2$ + c . . . must remain bounded. However, although the points of the Mandelbrot set are defined in a complex plane, it has been emphasized that the Mandelbrot set lies in the plane of parameter values (c or μ) rather than the plane in which z values orbit (see for example Peitgen et al. 845). Elert provides the following further clarification: "One way to picture Mandelbrot and Julia sets is as complex ordered pairs (c, z) such that the mapping f: z --> z^2 + c does not escape to infinity when iterated. Julia sets are slices parallel to the z-axis while the Mandelbrot set is a slice along the c-axis through the origin. As the coordinate system is complex, however, these 'axes' are actually planes. The Mandelbrot and Julia sets are therefore two-dimensional cross-sections through a four-dimensional parent set; the mother of all iterated quadratic mappings so to speak." (Elert 23.shtml) Interesting animation which demonstrates the transition between "successive slices through the four-dimensional mother set as we shift the cross-sectional plane from the z = 0 plane of the Mandelbrot set to the c = -1 plane of a particular Julia set (the San Marco Dragon)" has been provided (Elert movies/hybrid.mov).

Delimiting Values for the Sets

Recall the threshold radius r(c) = max(|c|,2) described above: when an orbit yields any z_k > r(c), this orbit will escape toward infinity. Note that r(c) can take on values greater than 2 when |c| > 2, though these sets will be disconnected. In contrast, the values of |c| for points in the Mandelbrot set must always be less than or equal to 2. On the real axis, the interval [-2, 0.25] consists entirely of points in M, and only for c=-2+0i can |c| = 2 (Peitgen et al. 845). To state also the obvious, for Julia sets having interior prisoner points, all points in the set lie on the boundary (i.e., Julia sets do not include interior points) whereas the points of the Mandelbrot include the interior points (i.e., the main cardioid and its buds often depicted in solid black).

The Cardioid Region and Major Buds (Atoms) of M

M consists of a cardioid or heart-like main body extending on the real axis from -0.75 to 0.25 and numerous "buds", the largest of which extends to the left of -0.75, again on the real axis. Mandelbrot terms these buds "atoms" (Mandelbrot 183). In the cardioid region, one of the fixed points is attracting (and is the point to which interior prisoner points converge) while the other is in the Julia set and is repelling (Peitgen et al. 855). Peitgen et al. provide a pair of equations parameterized on the parameter phi which generate this cardioid:

$$x = 0.5\cos(phi) - 0.25\cos(2phi)$$
$$y = 0.5\sin(phi) - 0.25\sin(2phi).$$

For example, the right margin of the cardioid is given for phi = 0 as 0.25+0i. When the parameter phi takes on values 2pi/k for integer values k = 2,3,4,5,6,..., there is a bud extending outward from the cardioid. The main or largest bud arises for phi = 2pi/2 (180°) and touches the cardioid at only one point. This type of connection is apparently the case for other buds. This bud is a perfectly circular disk of radius 0.25 centered at c=-1+0i (Peitgen et al. 862). Other buds arises at phi = 2pi/3 = 120° and 2pi/4 = 90°. The parameter phi does not correspond to the usual polar angle for the complex plane.

For example, the values corresponding to phi = pi/2=90° are x = 0.25 and y = 0.5 whereas one would expect x to be 0 for complex argument = 90°. Moreover, the second largest bud (for which phi is 120° arises at -0.125+0.64952i) visually appears to arise at a little more than 90° in the complex plane with respect to the origin. Although Peitgen et al. do not explicitly state this, it appears that buds also arise at values of phi = 2pi(1 + n)/k for values of n = 0, 1, 2, ..., thus providing the matching buds by symmetry. For example, the second largest bud, at phi = 120°, has a matching counterpart at phi = 240°.

The Main Bud Corresponds to Period-2 Julia Sets

For any c lying within the largest bud ($|1 + c| < 1/4$), Peitgen et al. (863) demonstrate that the two fixed points of the Julia set are both repelling. One of the points iterates to infinity. The orbit of points starting near the other fixed point, after adequate iterations, stabilizes to an oscillation between two values (similar to the period-2 regime of the Feigenbaum bifurcation diagram). These "period" limit points do not correspond to either of the "fixed" points. Alligood et al (169) refer to the interior prisoner points of a typical period-3 Julia set as a basin of a "period-three sink". By comparison, the cardioid region of M corresponds to Julia sets having periodicity = 1 and in which the period-1 limiting point is the same as one of the fixed points. The disk of the largest bud is thus said to be the period-two disk. For example, when c=-1+0i, any prisoner point of the Julia set evolves to oscillate between 0+0i and 1+0i. This behavior can easily be confirmed in Excel. When c is other than -1 but in the period-2 disk, the period-2 alternating values are not usually integral. For example, for c = -0.75+0i, the orbit of the starting point -1+0i oscillates between about -0.531 and -0.468. All the prisoner points of this Julia set are attracted to this same period-2 orbit.

Other Buds of M Correspond to Julia Sets of Higher Periodicity

Similarly, the values of c within the next largest buds of M correspond to Julia sets with period-3 orbits for prisoner points, and other buds are associated with higher degrees of periodicity.

This can be tested in the Excel application for example by selecting c=-0.12+0.74i, which lies in the large period 3 bud of M (arising at phi = 120°). After the orbit stabilizes (i.e., after 500 iterations or so), the orbital values are seen to cycle between -0.65306+0.56253i, -0.00995+0.00526i, and -0.11992+0.73989i. Other buds correspond to Julia sets with higher orbital periodicities. Peitgen et al. (866) state in essence the following: if two buds $b_1$and b_3 have periodicities p and q, then the periodicity of the largest bud b_2 that is smaller than both b_1 and b_3 and which lies on the cardioid between them is equal to p + q.

Indifferent Fixed Points and the Siegel Disk

Peitgen et al. (866) state that "indifferent" fixed points, which are neither attractive or repellent, arise for values of c right on the cardioid including at the touching points of buds. For some values of c corresponding to a parameter angle phi = 2(pi)m where m is irrational (such as -0.3905407802-0.5867879073i), the corresponding Julia prisoner set contains a disk called a Siegel disk (Peitgen et al. 868). The fixed point (-0.368684394 - 0.3377451165i in this case) is termed "irrationally indifferent" and the orbits of points near this fixed point within the Siegel disk rotate about the point but do not converge toward it.

Dendritic Julia Sets

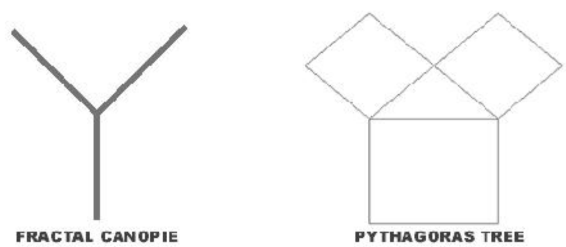
Some connected Julia sets consist of curves with no interior points. The most trivial example is c=-2+0i, for which the Julia set is a straight line [-2, 0]. Other values of c such as c=0+i produce a dendritic pattern again with no interior points. Other Julia sets may contain both dendritic segments lacking interior points and regions with interior points, e.g., c = -1.77578 (Peitgen et al. 873).

Mandelbrot Set Local Appearance is Mirrored in the Corresponding Julia Sets

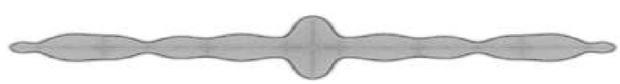
The Mandelbrot set serves as a roadmap to or table of contents for the Julia sets (Peitgen et al. 855-895). To varying degrees, but in some cases quite striking, there is a correspondence or quasi-similarity between the appearance of portions of the Mandelbrot set and the Julia sets corresponding to the c values in that region of M. For example, at the so-called Misiurewicz points such as 0+i, the M and J_c sets become asymptotically similar as the degree of magnification increases and after suitable scaling and rotational adjustment (Peitgen et al. 887-8). This property of asymptotic similarity at Misiurewicz points was proven by Tan Lei (Peitgen et al. 889). A web-based interactive demonstration of quasi-similarity is found at the Julia and Mandelbrot Set Explorer website (Joyce; access restricted to academic proxy connection). An especially detailed collection of color images (with black and white thumbnail diagrams) showing the mapping of the M set to Julia sets can be found at the website "What is the Mandelbrot Set?" (Brin).

Pythagoras Trees

Pythagoras trees are basically the same as **fractal canopies**, only they use rectangles and triangles instead of splitting lines:

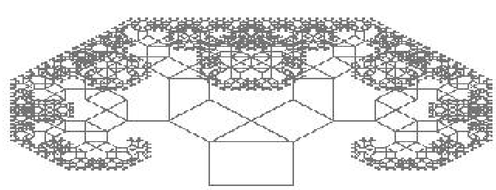


This allows us to give fractal trees a width, which is obviously better for modeling **plants**.

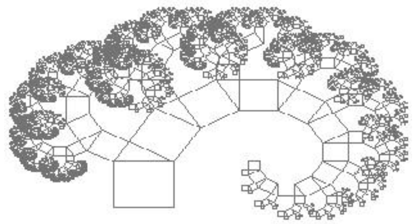


Different Kinds of Trees

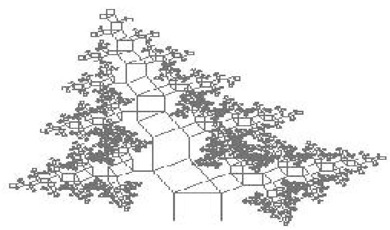
By using squares and 45-45-90 triangles, we can create this fractal:



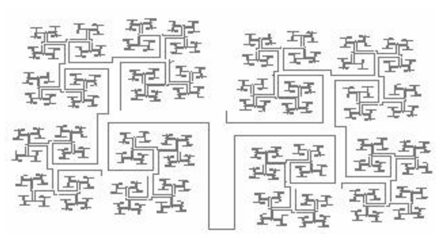
By using a 30-60-90 triangle instead, we can make this tree bend on one side, which creates a lopsided Pythagoras tree:



By doing the same, but flipping the triangle every time we can get rid of this create Pythagoras Christmas tree:

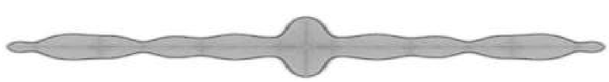


Using triangles allows us to create the angles we need between the "branches". However, if you are familiar with **fractal canopies** you probably know that one of them, the **H-fractal** uses a 180 degree angle between the branches. Its Pythagoras tree equivalent is the **Mandelbrot Tree**, which does not use triangles:



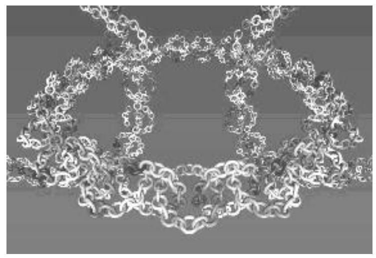
Nonstandard Fractals

Our website uses a system in which we categorized all fractals into 14 **types**. Into the category of "nonstandard fractals" we grouped several fractals that did not fit into any of these. Obviously, it is not a true mathematical term.

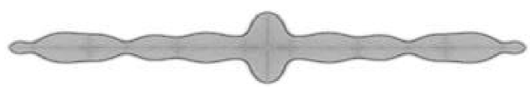


Chain Fractal

Consider starting with a simple ring. Substitute this ring with a chain that has 20 rings, connected to each other in a circle. Now substitute each of those 20 rings with a chain. Then substitute each of the 400 rings with a chain, and continue **iterating** these substitutions. What you get is this fractal:

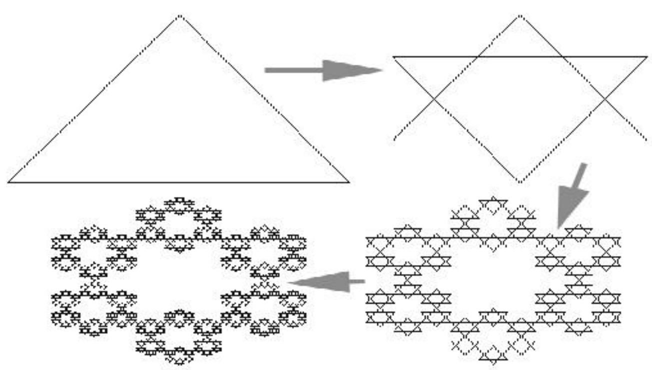


If you are familiar with knots, this kind of figure is called a link. The substitution of a figure with another figure is used **generator iteration** and is used in **base-motif fractals**. However, we considered this fractal nonstandard because, unlike base-motif fractals, it uses circles instead of line segments.



"Star of David" Fractal

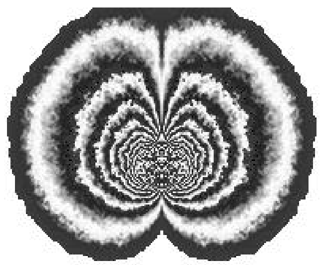
This fractal was created by our team and later turned out not to fit into any categories as well. It is also based on **generator iteration** where the figure substituted is not a line segment. In this fractal we start with an equilateral triangle. At every step of **iteration** we put a flipped triangle on top of every triangle. In other words, we substitute every triangle with a Star of David:

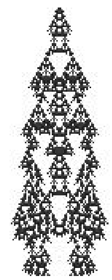


The fractal we get at the end is what we called the Star of David Fractal

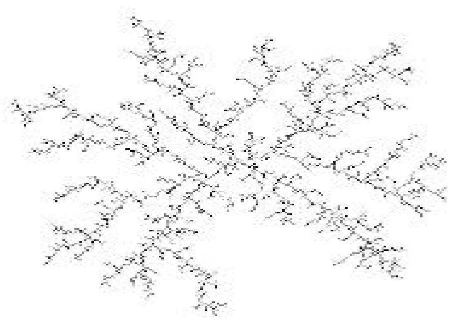
Cellular Automata Fractals

Cellular automata is a system which uses a set of rules to simulate the growth of cells or organisms. Perhaps the most famous cellular automata which you might know is the game of Life. Sometimes, cellular automata can create patterns which turn out to be fractal. In fact, fractal cellular automata are very important in simulating the growth of **bacteria**. The pictures below are examples of cellular automata which are fractal in shape:



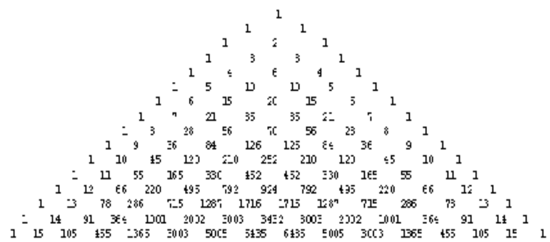


The most famous fractals of this type are probably the *diffusion fractals*, in which the spreading occurs from the center outwards. These fractals are also extremely useful in studying various kinds of **diffusion**:

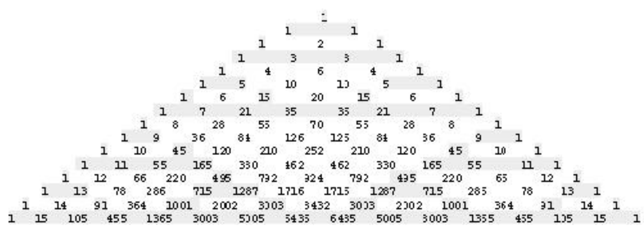


Pascal's Traingle

Form a pattern of numbers by making a triangle, in which every number is the sum of the two numbers above it. What you get is the *Pascal's Triangle*, which is very famous – only not to fractals.



However, if we color all odd numbers in this triangle a different color.... we get a fractal!



Indeed, the pattern formed is the same as the one in the **Sierpinski Triangle**. Isn't it amazing with all the places where you can find fractals?

Conclusion

The mathematical behavior and graphical intricacies of the Julia sets provide fine examples of the fascinating world of strange attractors. We have enjoyed taking this opportunity to examine these sets. However, many things in nature are described better with dimension being part of the way between two whole numbers. While a straight line has a dimension of exactly one, a fractal curve will have a dimension between one and two, depending on how much space it takes up as it curves and twists. The more a fractal fills up a plane, the closer it approaches two dimensions. In the same manner of thinking, a *wavy* fractal scene will cover a dimension somewhere between two and three. Hence, a fractal landscape which consists of a hill covered with tiny bumps would be closer to two dimensions, while a landscape composed of a rough surface with many average sized hills would be much closer to the third dimension.

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